AN ANALOGUE OF GREENBERG'S PSEUDO-NULL CONJECTURE FOR CM FIELDS

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Abstract. Let K be a CM field, and let K^+ be its maximal totally real subfield. Assume that the primes above p in K^+ split in K. Let S be a set containing exactly half of the prime ideals of K lying above p. We show that, assuming Leopoldt's conjecture holds for K and p, there exists a unique $\mathbb{Z}_p$-extension of K that is unramified outside S (the S-ramified $\mathbb{Z}_p$-extension of K). Such $\mathbb{Z}_p$-extensions for CM fields exhibit properties similar to the cyclotomic $\mathbb{Z}_p$-extensions of totally real fields. For example, Greenberg established a criterion ensuring that the Iwasawa invariants satisfy $\mu = \lambda = 0$ for the cyclotomic $\mathbb{Z}_p$-extension of a totally real field. We prove analogous results for the S-ramified $\mathbb{Z}_p$-extension of a CM field. Additionally, we provide a numerical criterion for $\mu = \lambda = 0$ in the case of an imaginary biquadratic field, which parallels the criterion given by Fukuda and Komatsu for real quadratic fields. We also give a generalization of de Shalit's result regarding the index of unit groups.

1. Introduction

Let F be any number field and p be an odd prime. Let $F \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots \subset F_\infty = \bigcup_n F_n$ be a $\mathbb{Z}_p$-extension of F, that is, $\mathrm{Gal}(F_n/F) = \mathbb{Z}/p^n\mathbb{Z}$ and $\mathrm{Gal}(F_\infty/F) = \mathbb{Z}_p$. Let A_n be the p primary part of the class group of F_n. Iwasawa [Iwa73] proved that there are three integers λ, μ, and ν such that

$$|A_n| = p^{\mu p^n + \lambda n + \nu}$$

when n is sufficiently large. These are the so called Iwasawa invariants for the $\mathbb{Z}_p$-extension F_∞/F.

For each number field F, there is one obvious $\mathbb{Z}_p$-extension. Let ζ_{p^n} be the primitive p^n-th root of unity. Then $\mathbb{Q}(\zeta_{p^{n+1}})/\mathbb{Q}$ has a unique degree p^n sub-extension of $\mathbb{Q}$ denoted as $\mathbb{Q}_n$. Put $\mathbb{Q}_\infty = \bigcup_n \mathbb{Q}_n$ and $F_\infty = F\mathbb{Q}_\infty$. Then F_∞/F is a $\mathbb{Z}_p$-extension of F which is called the cyclotomic $\mathbb{Z}_p$-extension of F. Let F_n be the n-th layer of the $\mathbb{Z}_p$-extension of F_∞/F.

We recall Greenberg's conjecture [Gre76] here.

Conjecture 1.1 (Greenberg)**.** *If F is a totally real field, then $\mu = \lambda = 0$ for the cyclotomic $\mathbb{Z}_p$-extension of F.*

In other words, the order of the groups A_n should be bounded as $n \to \infty$ for the cyclotomic $\mathbb{Z}_p$-extension of a totally real field. Much research has been done to study this well-known conjecture, for example, [FK86a][Tay96][McC01].

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In this paper, we study a special $\mathbb{Z}_p$-extension of CM fields constructed in Section 2. Our goal is to illustrate that this special $\mathbb{Z}_p$-extension behaves similarly to the cyclotomic $\mathbb{Z}_p$-extension of totally real fields. We begin by presenting some results analogous to those in Greenberg's paper [Gre76] in Section 3 and Section 4. Given that Greenberg's conjecture has been extensively studied over the years, we establish further analogous results in the remaining sections, which is the most innovative part of the paper.

1.1. **Main Results.** Now, instead of considering the cyclotomic $\mathbb{Z}_p$-extension of a totally real field, we will consider a certain $\mathbb{Z}_p$-extension of a CM field. Throughout the paper p will be an odd prime. Let K be a CM field and K^+ be the maximal subfield fixed by complex conjugation. Let $S^+ = \{\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_s\}$ be a set containing primes of K^+ above p, and assume that $\mathcal{P}_i$ splits in K as $\mathcal{P}_i\mathcal{O}_K = \mathfrak{P}_i\tilde{\mathfrak{P}}_i$ for $1 \leq i \leq s$, where $\tilde{\mathfrak{P}}_i$ is the complex conjugation of $\mathfrak{P}_i$. We write

$$S = \{\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_s\}.$$

In Section 2, we prove the following theorem (which is Theorem 2.1 in the Section 2).

Theorem 1.2. *Assume that primes above p in K^+ split in the CM field K. Then there is a $\mathbb{Z}_p$-extension K_∞/K unramified outside S. If the Leopoldt's conjecture holds, then such $\mathbb{Z}_p$-extension is unique.*

We will refer to the $\mathbb{Z}_p$-extension K_∞/K in Theorem 1.2 as the S-ramified $\mathbb{Z}_p$-extension of K. The theorem is known in general cases by class field theory [Fuj17, Lemma 1]. For a proof using adelic language, see [Mai02, Theorem 5]. The theorem can also be viewed as an analogue of the fact that a totally real field has a unique $\mathbb{Z}_p$-extension [Was97, Theorem 13.4]. We provide a proof by following the approach used in Theorem 13.4 of Washington [Was97]. The idea is similar to the previously mentioned results, with minor differences in detail. We include this proof as it serves as the starting point of our research and provides a useful comparison.

The S-ramified $\mathbb{Z}_p$-extension of a CM field K is similar to the cyclotomic $\mathbb{Z}_p$-extension of K^+ in certain senses. For instance Fukuda and Komatsu [FK02],[FK14] give numerical evidence that the λ-invariant vanishes for S-ramified $\mathbb{Z}_p$-extensions of imaginary quadratic fields. Hence, we propose the following analogue of Greenberg's conjecture.

Conjecture 1.3. *Let K_∞/K be the S-ramified $\mathbb{Z}_p$-extension defined as Theorem 2.1. Then A_n will be bounded as $n \to \infty$. In other words, the Iwasawa invariant $\mu = \lambda = 0$.*

While we can not prove this conjecture, we can show that the conjecture holds under some assumptions. From now on, we assume that p is an odd prime and primes above p in K^+ split in K and all primes in S are totally ramified in the S-ramified $\mathbb{Z}_p$-extension. They correspond to the Assumption (2), (3), and (4) in Section 2.

Let $i_{n,m} : \mathrm{Cl}(K_n) \to \mathrm{Cl}(K_m)$ for $m \geq n$ be the natural map between class groups induced by inclusion. Let $H_n = \cup_{m \geq n} \mathrm{Ker}\ (i_{n,m})$.

Theorem 1.4. *Assume that p is inert in $K^+/\mathbb{Q}$ and Leopoldt's conjecture holds for K. Then the following are equivalent.*

(a) $A_0 = H_0$.

(b) $|A_n|$ *is bounded as* $n \to \infty$.

This is Theorem 4.1 in Section 4. It is analogous to Greenberg's Theorem 1 in [Gre76]. The above theorem is also contained in a more general statement [Min86, Prop, 1.B.]. The proof follows a similar approach to Greenberg's result[Gre76, Theorem 1]. We list it in the paper for comparison.

Let B_n be the subgroup of A_n fixed by $\mathrm{Gal}(K_n/K)$. Let D_n be the subgroup of A_n generated by prime ideals above S.

Theorem 1.5. *Assume* p *splits completely in* $K^+/\mathbb{Q}$ *and Leopoldt's conjecture holds for* K. *Then the following are equivalent.*

(a) $B_n = D_n$ *for all sufficiently large* n.

(b) $|A_n|$ *is bounded as* $n \to \infty$.

This is Theorem 4.3 in Section 4. It is analogous to Greenberg's Theorem 2 in [Gre76]. The above theorem was proved by Goto [Got06] when K is abelian CM field. Here we do not assume that K is abelian. Although the idea of proof is similar to Greenberg's [Gre76, Theorem 2], we include a proof for completeness in the paper. For a number field L, let $\mathcal{O}_L$ denote its ring of integers and $\mathcal{O}_L^\times$ its unit group. Throughout, put $K_0 = K$ and $\mathcal{O}_n := \mathcal{O}_{K_n}$ for $n \geq 0$, so that the unit group of K_n is $\mathcal{O}_n^\times$ and $\mathcal{O}_0^\times = \mathcal{O}_K^\times$. Let K be a CM field. We next compare S-ramified $\mathbb{Z}_p$-extension K_∞/K and cyclotomic $\mathbb{Z}_p$-extension K_∞^+/K^+. Let K_n be the nth layer of K_∞/K and K_n^+ be the nth layer of K_∞^+/K^+. Let N_n be the norm map from field K_n to K or K_n^+ to K^+.

Proposition 1.6. *Assume* p *splits completely in* $K/\mathbb{Q}$ *and Leopoldt's conjecture holds for* K.

$$\mathcal{O}_K^\times/(N_n(K_n^\times) \cap \mathcal{O}_K^\times) \cong \mathcal{O}_{K^+}^\times/(N_n((K_n^+)^\times) \cap \mathcal{O}_{K^+}^\times)$$

This is Proposition 5.1 in Section 5. The proposition is interesting because the extensions K_n/K and K_n^+/K^+ are globally unrelated, but locally similar.

There are various kinds of numerical criterion to determine when $\mu = \lambda = 0$ for the cyclotomic $\mathbb{Z}_p$-extension of a real quadratic field. We give a similar numerical criterion to that of Fukuda and Komatsu [FK86a] for the S-ramified $\mathbb{Z}_p$-extension of an imaginary biquadratic field. Let $m, d \in \mathbb{Z}^+$ that are squarefree and coprime. Denote $k = \mathbb{Q}(\sqrt{-m})$, $K^+ = \mathbb{Q}(\sqrt{d})$, $K = K^+k$, and ε to be the fundamental unit for K. Suppose that p splits completely in K, with $p\mathcal{O}_k = \mathfrak{p}\tilde{\mathfrak{p}}$ and $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}\bar{\mathfrak{P}}$. Take $S = \{\mathfrak{P}, \bar{\mathfrak{P}}\}$. The following is Theorem 5.4 in Section 5.

Theorem 1.7. *Suppose* p *doesn't divide the class number of* K, *and that* r *is the largest positive integer such that*

$$\varepsilon^{p-1} \equiv 1 \pmod{\bar{\mathfrak{P}}^r}.$$

If $N_{r-1}(\mathcal{O}_{r-1}^\times) = \mathcal{O}_0^\times$, *then* $\mu = \lambda = 0$ *for the* S*-ramified* $\mathbb{Z}_p$*-extension of* K_∞/K.

Lastly, we generalize a result from Ehud de Shalit's work [Sha89, Theorem 2]. Let e be the smallest integer such that K_∞/K is totally ramified at all primes in S. Denote by $\mathrm{N}_{m,n}$ the norm map from K_m to

K_n for $m \geq n$. For $n \geq e$, define

$$E_n^0 = \{\beta \in \mathcal{O}_n^\times \mid \mathrm{N}_{n,e}(\beta) = 1\},$$
$$E_n' = \bigcap_{m \geq n} \mathrm{N}_{m,n}(E_m^0).$$

Here E_n^0 and E_n' denote subgroups of $\mathcal{O}_n^\times$, and $E_n' \subset E_n^0$. Recall that A_n denotes the p-primary part of the class group of K_n. The following is Theorem 6.1 in Section 6.

Theorem 1.8.

(a) For sufficiently large n, the index $[E_n^0 : E_n']$ divides $|A_n|$.

(b) For sufficiently large n, every element $e_n \in E_n'$ appears in a norm-coherent sequence $(e_m) \in \varprojlim \mathcal{O}_n^\times$.

1.2. **Structure of the paper.** In Section 2, we prove Theorem 2.1. In Section 3, we prove some general properties for S-ramified $\mathbb{Z}_p$-extension and introduce some assumptions. In Section 4, we first list the case when p is inert in $K^+/\mathbb{Q}$ and then we deal with the case when p splits completely in $K^+/\mathbb{Q}$. In Section 5, we compare ambiguous class group between S-ramified $\mathbb{Z}_p$-extension of K and cyclotomic $\mathbb{Z}_p$-extension of K^+. We also give a numerical criterion for $\mu = \lambda = 0$ for biquadratic fields at the end of the section. In section 6, we prove a generalization of de Shalit's theorem.

The main goal of this paper is to demonstrate that the cyclotomic $\mathbb{Z}_p$-extension of a totally real field F behaves in a manner analogous to the S-ramified $\mathbb{Z}_p$-extension of a CM field K. In Sections 2, 3, and 4, we collect several propositions concerning the S-ramified $\mathbb{Z}_p$-extension of a CM field K that parallel well-known properties of the cyclotomic $\mathbb{Z}_p$-extension of a totally real field F. To make this analogy transparent, we deliberately adopt arguments similar to those used in the cyclotomic setting. Since many of the statements resemble classical results and the methods are standard, experts may choose to focus primarily on the statements and skip the proofs to grasp the overall picture.

The most novel contribution of this paper appears in Section 5. To the best of our knowledge, we are the first to establish a formula that directly relates the S-ramified $\mathbb{Z}_p$-extension and the cyclotomic $\mathbb{Z}_p$-extension; see, for instance, Proposition 5.1. Finally, the main objective of Section 6 is to show that the unit groups in these two types of $\mathbb{Z}_p$-extensions exhibit analogous behavior. While most existing works focus on comparing the behavior of class groups in these two settings, we emphasize that their unit structures also display striking similarities.

1.3. **Potential Directions.** Here we remark on some possible future directions one might take in studying Conjecture 1.3:

(i) Greenberg [Gre01] also proposed the "Generalized Greenberg Conjecture".

Conjecture 1.9. *[Gre01, Conjecture 3.5] Suppose that F is any number field and p is a prime. Let $\tilde{F}$ denote the composition of all $\mathbb{Z}_p$-extensions of F. Let $\tilde{L}$ denote the pro-p Hilbert class field of $\tilde{F}$ and let $\tilde{X} = Gal(\tilde{L}/\tilde{F})$, regarded as a module over the ring $\tilde{\Lambda} = \mathbb{Z}_p[[Gal(\tilde{F}/F)]]$. Then $\tilde{X}$ is a pseudo-null $\tilde{\Lambda}$-module.*

What is the relationship between Conjecture 1.3 and the Generalized Greenberg Conjecture? Could Conjecture 1.3 be a special case of the Generalized Greenberg Conjecture? We make a wild guess that

Conjecture 1.1, Conjecture 1.3, and Conjecture 1.9 are all equivalent. While we do not have rigorous evidence, our intuition is as follows:

- The Conjecture 1.1 and Conjecture 1.3 exhibit similar behavior, as demonstrated in this paper.
- Conjecture 1.3 implies Conjecture 1.9 under certain conditions, as shown in [Fuj17, Theorem 3].
- The Conjecture 1.1 is a special case of Conjecture 1.9.

This suggests a circular implication: the Conjecture 1.1 implies Conjecture 1.3, which in turn implies Conjecture 1.9, and finally, Conjecture 1.9 implies the Conjecture 1.1.

(ii) Fukuda and Komatsu [FK02] [FK14] only calculate the examples for the S-ramified $\mathbb{Z}_p$-extension of K_∞/K defined in Theorem 2.1 for imaginary quadratic fields K. All calculated examples have $\lambda = 0$. Is it possible to calculate more examples for the general CM field K?

(iii) In [Gil87] and [Sch87], the authors proved that $\mu = 0$ for such S-ramified $\mathbb{Z}_p$-extension of K_∞/K when K is imaginary quadratic field. Ferrero-Washington [FW79] proved that $\mu = 0$ for the cyclotomic $\mathbb{Z}_p$-extension of an abelian number field. Could one adapt their argument to prove that $\mu = 0$ for S-ramified $\mathbb{Z}_p$-extension when the CM field K is abelian?

(iv) The S-ramified $\mathbb{Z}_p$-extension appears to have greater potential in Iwasawa theory for abelian varieties, as it can be constructed from geometric considerations; see [Sha89] for work of Ehud de Shalit. For example, in [Sha87], the S-ramified $\mathbb{Z}_p$-extension of an imaginary quadratic field K is employed, although it is not essential for the main results of that paper. In contrast, in the recent work [FW22], the use of the S-ramified $\mathbb{Z}_p$-extension of an imaginary quadratic field is indispensable for establishing their results on the Iwasawa main conjecture.

2. Uniqueness of S-Ramified $\mathbb{Z}_p$-Extensions

Let K be a CM field and K^+ be the maximal subfield fixed by complex conjugation. Let $S^+ = \{\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_s\}$ be the set of primes of K^+ above p. Assume that each of the primes above p in K^+ split in K. Write $\mathcal{P}_i\mathcal{O}_K = \mathfrak{P}_i\tilde{\mathfrak{P}}_i$ for $1 \leq i \leq s$, where $\tilde{\mathfrak{P}}_i$ is the complex conjugation of $\mathfrak{P}_i$, and set $S = \{\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_s\}$. The following theorem can be viewed as the analogue of the fact that there is a unique $\mathbb{Z}_p$-extension of a totally real field for which Leopoldt's conjecture holds. The method of the proof is similar to the proof of Theorem 13.4 in Washington [Was97].

Theorem 2.1. *Let T be the maximal abelian extension of K unramified outside S. Then there is a surjective homomorphism $Gal(T/K) \to \mathbb{Z}_p^{1+\delta}$ with finite kernel, where δ is the Leopoldt defect (see [Was97, Theorem 13.4 page 266]). In particular, if Leopoldt's conjecture holds for K, then $\delta = 0$ and there is a unique $\mathbb{Z}_p$-extension contained in T.*

Proof. Let T be the maximal abelian extension of K which is unramified outside S. Let $\mathbb{A}_K^\times$ be the group of idèles of K. By class field theory, there is a closed subgroup $R \subset \mathbb{A}_K^\times$ such that

$$\mathrm{Gal}(T/K) \cong \mathbb{A}_K^\times / R$$

Let U_v be the local unit group at a place v of K and $U_v = K_v^\times$ if v is an archimedean place. Define

$$U' = \prod_{i=1}^{s} U_{\mathfrak{P}_i}, \qquad U'' = \prod_{v \notin S} U_v, \qquad U = U' \times U''$$

We can view U' as a subgroup of $\mathbb{A}_K^\times$ by placing a 1 in each component outside of S, and we can view U'' as a subgroup of $\mathbb{A}_K^\times$ in a similar way. Let $W = \overline{K^\times U''}$ be the closure of $K^\times U''$ inside $\mathbb{A}_K^\times$. Since T is unramified outside S, we have $W \subset R$. Since T is maximal, we must have $W = R$. Thus $\mathrm{Gal}(T/K) \cong \mathbb{A}_K^\times / W$. Let H be the Hilbert class field of K. Then a similar argument shows

$$\mathrm{Gal}(H/K) = \mathbb{A}_K^\times / (K^\times U).$$

Therefore, we have $\mathrm{Gal}(T/H) = K^\times U/W = U'W/W \cong U'/(U' \cap W)$. Let $U_{1,\mathfrak{P}_i}$ be the group of local units congruent to 1 modulo $\mathfrak{P}_i$, and put $U_1 = \prod_{i=1}^s U_{1,\mathfrak{P}_i}$. Then

$$U' = U_1 \times (\text{finite group})$$

Hence

$$\mathrm{Gal}(T/H)/(\text{finite group}) \cong U_1(U' \cap W)/(U' \cap W) = U_1/(U_1 \cap W)$$

Let $E_S(K)$ be the group of units in K congruent to 1 modulo the primes in S. Then we can embed $E_S(K)$ in $\mathbb{A}_K^\times$ by

$$\varphi : E_S(K) \hookrightarrow U_1 \subset \mathbb{A}_K^\times.$$

In a moment we will prove Lemma 2.2, which implies

$$U_1 \cap W = U_1 \cap \overline{K^\times U''} = \overline{\varphi(E_S(K))}.$$

Writing $E_{S^+}(K^+)$ to be the group of units in K^+ congruent to 1 modulo the primes in S^+, we have

$$E_{S^+}(K^+) \subset E_S(K) \subset \mathcal{O}_K^\times.$$

Since $\mathrm{Rank}_{\mathbb{Z}} \mathcal{O}_K^\times = \mathrm{Rank}_{\mathbb{Z}} E_{S^+}(K^+) = [K : \mathbb{Q}]/2 - 1$, the index of $E_{S^+}(K^+)$ in $E_S(K)$ is finite. Hence, the index of $\overline{E_{S^+}(K^+)}$ in $\overline{E_S(K)}$ is finite. Assume that $\mathrm{Rank}_{\mathbb{Z}_p}(\overline{E_{S^+}(K^+)}) = [K : \mathbb{Q}]/2 - 1 - \delta$ and $\delta \geq 0$. Then

$$\overline{\varphi(E_S(K))} \cong \mathbb{Z}_p^{[K:\mathbb{Q}]/2-1-\delta} \times (\text{finite group}). \tag{1}$$

Recall that [Was97, Page 75] Leopoldt's conjecture predicts that $\delta = 0$. By [Was97, Prop 5.7], we know

$$U_1 \cong (\text{finite group}) \times \mathbb{Z}_p^{[K:\mathbb{Q}]/2}.$$

Therefore

$$U_1/(U_1 \cap W) = (\text{finite group}) \times \mathbb{Z}_p^{1+\delta}.$$

Hence

$$\mathrm{Gal}(T/H) = (\text{finite group}) \times \mathbb{Z}_p^{1+\delta}.$$

Since $\mathrm{Gal}(H/K) \cong \mathrm{Cl}(K)$ is a finite group,

$$\mathrm{Gal}(T/K)/\mathbb{Z}_p^{1+\delta} \cong (\text{finite group}).$$

We will also prove Lemma 2.3, which shows there is a finite group such that

$$\mathrm{Gal}(T/K)/(\text{finite group}) \cong \mathbb{Z}_p^{1+\delta}.$$

Let $\tilde{K}$ be the compositum of all $\mathbb{Z}_p$-extension of K unramified outside S. The fixed field of this finite group must be $\tilde{K}$ and $\mathrm{Gal}(\tilde{K}/K) \cong \mathbb{Z}_p^{1+\delta}$. If Leoplodt's conjecture holds for K, then Leoplodt's conjecture holds for K^+. Hence $\delta = 0$, which implies there is a unique $\mathbb{Z}_p$-extension of K unramified outside S.

□

Lemma 2.2. $U_1 \cap W = U_1 \cap \overline{K^\times U''} = \overline{\varphi(E_S(K))}$

Proof. Take $\varepsilon \in E_S(K)$. Then $\varphi(\varepsilon) \in U_1$ and

$$\varphi(\varepsilon) = \varepsilon \frac{\varphi(\varepsilon)}{\varepsilon}.$$

We have $\varepsilon \in K^\times$ and $\varphi(\varepsilon)/\varepsilon \in U''$. Hence $\overline{\varphi(E_S(K))} \subset U_1 \cap W$.

Recall that in a topological space, the closure generated by a set V is the intersection of closed subsets containing V. Define $U_{n,\mathfrak{P}_i} = \{x \in U_{\mathfrak{P}_i} | x \equiv 1 \mod \mathfrak{P}_i^n\}$ and $U_n = \prod_{i=1}^s U_{n,\mathfrak{P}_i}$. Then

$$W = \overline{K^\times U''} = \bigcap_n K^\times U'' U_n$$

$$\overline{\varphi(E_S(K))} = \bigcap_n \varphi(E_S(K)) U_n$$

It suffices to show that

$$U_1 \cap K^\times U'' U_n \subset \varphi(E_S(K)) U_n$$

Take $x \in K^\times$, $u'' \in U''$ and $u \in U_n$ such that $xu''u \in U_1$. Then $xu'' \in U_1$. At primes of S, we have $u'' = 1$, so x is a local unit. At the primes outside of S, we have that u'' is a unit, hence x is also a local unit. Therefore, x is a local unit everywhere which implies x is a global unit. At the primes of S we have $xu'' = x$, and at the primes outside of S we have $xu'' = 1$. This exactly means that $\varphi(x) = xu''$. Hence

$$xu''u \in \varphi(E_S(K)) U_n.$$

This completes the proof of the lemma. □

Lemma 2.3. *Let J be a profinite abelian group and $\mathbb{Z}_p^r \subset J$. Assume that $J/\mathbb{Z}_p^r$ is a finite group. Then there exists a finite group T such that*

$$J/T \cong \mathbb{Z}_p^r$$

Proof. Write $G = J/\mathbb{Z}_p^r$ and denote N to be the size of G. Then

$$N\mathbb{Z}_p^r \subset NJ \subset \mathbb{Z}_p^r$$

Hence $NJ \cong \mathbb{Z}_p^r$. Putting $J[N] = \{x \in J | Nx = 0\}$, we have

$$J/J[N] \cong NJ \cong \mathbb{Z}_p^r$$

$$x \to Nx$$

In fact, $\#J[N] \leq N$ is finite by the snake lemma.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & J[N] & \longrightarrow & G[N] & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_p^r & \longrightarrow & J & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} & & \\ 0 & \longrightarrow & \mathbb{Z}_p^r & \longrightarrow & J & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

□

Remark 2.4. Our proof follows a conventional approach in order to make the comparison with the cyclotomic $\mathbb{Z}_p$-extension of a totally real field more transparent. We retain this proof because Lemma 2.2 gives a clear description of $\overline{\varphi(E_S(K))}$. We further remark that Theorem 2.1 holds for $p = 2$ as well, though we only consider odd primes in the paper.

We thank the anonymous referee for pointing out that the argument can be simplified by considering T', the maximal abelian pro-p extension of K unramified outside S, instead of T, the maximal abelian extension of K unramified outside S. In this setting, one can directly apply [Jau98, §2], see also[Gra05], without using Lemma 2.2, to obtain

$$\mathrm{Gal}(T'/H') \simeq U_1/\overline{\varphi(E_S(K))},$$

where H' is the Hilbert p-class field of K. Equation (1) shows that the $\mathbb{Z}_p$-rank of $U_1/\overline{\varphi(E_S(K))}$ is $1 + \delta$. Hence, $\mathrm{Gal}(T'/K)$ also has $\mathbb{Z}_p$-rank $1 + \delta$.

Since $\mathrm{Gal}(T'/K)$ is an abelian pro-p group, the structure theorem for finitely generated abelian pro-p groups allows one to conclude directly, without using Lemma 2.3, that if $\widetilde{K}$ denotes the compositum of all $\mathbb{Z}_p$-extensions of K unramified outside S, then $\mathrm{Gal}(\widetilde{K}/K) \simeq \mathbb{Z}_p^{1+\delta}$.

3. General Properties for S-Ramified $\mathbb{Z}_p$-Extensions of CM Fields

Let K be a CM field. Throughout the paper, we will assume that

$$\text{each of the primes above } p \text{ in } K^+ \text{ split in } K \tag{2}$$

$$p \text{ is an odd prime.} \tag{3}$$

We will consider the $\mathbb{Z}_p$-extension K_∞/K unramified outside S which exists by Theorem 2.1. It is unique if Leopoldt's conjecture holds for K. We call this $\mathbb{Z}_p$-extension of the CM field K the S-ramified $\mathbb{Z}_p$-extension.

We will show that such S-ramified $\mathbb{Z}_p$-extensions K_∞/K for CM field K have similar properties as cyclotomic $\mathbb{Z}_p$-extensions of totally real fields. We will prove a series of results analogous to that of Greenberg's in [Gre76], the proofs of which closely follow the arguments found in there.

In the case of the cyclotomic $\mathbb{Z}_p$-extension of totally real fields, any prime above p will become totally ramified starting from some higher layer. We show similar property holds for S-ramified $\mathbb{Z}_p$-extension K_∞/K.

Lemma 3.1. *Assume that Leopoldt's conjecture holds for K and the odd prime p. Then every prime in S ramifies in the unique S-ramified $\mathbb{Z}_p$-extension K_∞/K.*

Proof. We use the notation and results from the proof of Theorem 2.1. By local class field theory, it suffices to show that, for every i, the image of $U_{1,\mathfrak{P}_i}$ in $U_1/(U_1 \cap W)$ has positive $\mathbb{Z}_p$-rank.

For $n \geq 1$, let $U_{n,\mathfrak{P}_i}$ denote the group of local units congruent to 1 modulo $\mathfrak{P}_i^n$. For n sufficiently large, the p-adic logarithm induces an isomorphism $\log\colon U_{n,\mathfrak{P}_i} \xrightarrow{\sim} \mathfrak{P}_i^n \mathcal{O}_{K_{\mathfrak{P}_i}}$ by [Neu99, Proposition II.5.5]. Since $U_{n,\mathfrak{P}_i}$ has finite index in $U_{1,\mathfrak{P}_i}$ and $\mathfrak{P}_i^n \mathcal{O}_{K_{\mathfrak{P}_i}}$ has finite index in $\mathcal{O}_{K_{\mathfrak{P}_i}}$, tensoring with $\mathbb{Q}_p$ gives

$$\begin{aligned} U_{1,\mathfrak{P}_i} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p &\cong U_{n,\mathfrak{P}_i} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ &\cong \mathfrak{P}_i^n \mathcal{O}_{K_{\mathfrak{P}_i}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ &\cong K_{\mathfrak{P}_i} \cong K^+_{\mathcal{P}_i}, \end{aligned}$$

where $\mathcal{P}_i$ is the prime of K^+ below $\mathfrak{P}_i$. The last isomorphism follows from the assumption that $\mathcal{P}_i$ splits in K/K^+. Thus $U_1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \bigoplus_{i=1}^s K^+_{\mathcal{P}_i}$. Under this identification, consider the $\mathbb{Q}_p$-linear map

$$\Theta\colon U_1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \mathbb{Q}_p, \qquad (x_{\mathcal{P}_i})_{i=1}^s \longmapsto \sum_{i=1}^s \mathrm{Tr}_{K^+_{\mathcal{P}_i}/\mathbb{Q}_p}(x_{\mathcal{P}_i}).$$

Let $\varepsilon \in E_{S^+}(K^+)$. Then

$$\begin{aligned} \Theta\big((\log_{\mathcal{P}_i} \varepsilon)_{i=1}^s\big) &= \sum_{i=1}^s \mathrm{Tr}_{K^+_{\mathcal{P}_i}/\mathbb{Q}_p}\big(\log_{\mathcal{P}_i} \varepsilon\big) \\ &= \log_p\big(\mathrm{N}_{K^+/\mathbb{Q}}(\varepsilon)\big) = 0, \end{aligned}$$

since $\mathrm{N}_{K^+/\mathbb{Q}}(\varepsilon) = \pm 1$. As $E_{S^+}(K^+)$ has finite index in $E_S(K)$, their images have the same $\mathbb{Q}_p$-span. Hence

$$\overline{\varphi(E_S(K))} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \subseteq \ker \Theta.$$

By Leopoldt's conjecture, $\dim_{\mathbb{Q}_p}\left(\overline{\varphi(E_S(K))} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right) = [K^+ : \mathbb{Q}] - 1$. On the other hand, $\dim_{\mathbb{Q}_p}\left(U_1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right) = [K^+ : \mathbb{Q}]$, and Θ is nonzero. Hence $\dim_{\mathbb{Q}_p} \ker \Theta = [K^+ : \mathbb{Q}] - 1$, so $\overline{\varphi(E_S(K))} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \ker \Theta$.

By Lemma 2.2, $\overline{\varphi(E_S(K))} = U_1 \cap W$, and Theorem 2.1 shows that $U_1/(U_1 \cap W)$ has $\mathbb{Z}_p$-rank one. Now fix i. The composite $U_{1,\mathfrak{P}_i} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow U_1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\Theta} \mathbb{Q}_p$ identifies with $\mathrm{Tr}_{K^+_{\mathcal{P}_i}/\mathbb{Q}_p}$, which is nonzero. Hence$U_{1,\mathfrak{P}_i} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \not\subseteq (U_1 \cap W) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Therefore, the image of $U_{1,\mathfrak{P}_i}$ in $U_1/(U_1 \cap W)$ has positive $\mathbb{Z}_p$-rank, and hence rank one. By local class field theory, this image corresponds to the inertia subgroup at $\mathfrak{P}_i$ in the unique S-ramified $\mathbb{Z}_p$-extension. Thus the inertia subgroup is nontrivial, and consequently $\mathfrak{P}_i$ ramifies in K_∞/K. □

Let K_n be the nth layer of the S-ramified $\mathbb{Z}_p$-extension K_∞/K. Let A_n denote the p-primary part of the class group of K_n, and let σ be a topological generator of $\Gamma := \mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$. Recall that for each prime $\mathcal{P}_i$ of K^+ we assume $\mathcal{P}_i \mathcal{O}_K = \mathfrak{P}_i \tilde{\mathfrak{P}}_i$ and denote $S = \{\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_s\}$. Following the notation of Greenberg, we define

$$B_n := \{c \in A_n \mid c^\sigma = c\} = A_n^\Gamma.$$

Here Γ acts on A_n through its quotient $\mathrm{Gal}(K_n/K)$. Greenberg [Gre76, Proposition 1] showed that $|B_n|$ is bounded for the cyclotomic $\mathbb{Z}_p$-extension of a totally real field assuming Leopoldt's conjecture. We have the following similar result.

Proposition 3.2. *Suppose that Leopoldt's conjecture holds for p and K. Then $|B_n|$ is bounded as $n \to \infty$ for the unique S-ramified $\mathbb{Z}_p$-extension of K.*

Proof. Let T be the maximal abelian extension of K unramified outside of S. By Theorem 2.1 we have that $\mathrm{Gal}(T/K_\infty) < \infty$. Let L'_n be the maximal pro-p abelian extension of K_n that is unramified over K_n. Then $A_n \cong \mathrm{Gal}(L'_n/K_n)$. Let L_n be the maximal pro-p abelian extension of K that is unramified over K_n, so $A_n^{\sigma-1} \cong \mathrm{Gal}(L'_n/L_n)$. Then $[L_n : K_n] = [A_n : A_n^{\sigma-1}] = |B_n|$. When n is large enough, K_∞/K_n is totally ramified. Thus $L_n \cap K_\infty = K_n$. Hence when n is large enough, we have $|B_n| = [L_n : K_n] = [L_nK_\infty : K_\infty] \leq [T : K_\infty] \leq \infty$. □

Let D_n be the subgroup of A_n which consists of ideal classes containing a product of prime ideals above the primes of S.

Remark 3.3. In [Gre76], Greenberg defines D_n to be the subgroup of A_n which consists of ideal classes that contain a product of prime ideals above p. In that situation, all primes above p are ramified in the cyclotomic $\mathbb{Z}_p$-extension. The difference in our case is that only the primes in S are ramified in the S-ramified $\mathbb{Z}_p$-extension.

Denote e to be the smallest positive integer such that the primes of K_e above S are totally ramified in K_∞/K_e.

Corollary 3.4. *Let $[\alpha] \in D_0$. Then the ideal α will become principal in K_m when m is sufficiently large.*

Proof. Take $\mathfrak{P}_i \in S$. Let $\Omega_{i,n}$ be the product of prime above $\mathfrak{P}_i$ in K_n. Then $\Omega_{i,n}\mathcal{O}_{K_m} = \Omega_{i,m}^{p^{m-n}}$ when $m \geq n \geq e$ since K_m/K_n is totally ramified at primes above S. Assume $\alpha = \prod_i \mathfrak{P}_i^{s_i}$ for some integer s_i. Then $\alpha\mathcal{O}_{K_n}$ is a product of $\Omega_{i,n}$. We have $\alpha\mathcal{O}_{K_m} = b_m^{p^{m-n}}$ for some ideal b_m in K_m and b_m is a product of $\Omega_{i,m}$. Hence, $b_m \in B_m$. By Proposition 3.2, we know B_m is bounded. Hence $\alpha\mathcal{O}_{K_m}$ become principal when m is large enough. □

The same argument can show the following result when K_∞/K is totally ramified at all primes in S.

Corollary 3.5. *Let $[\alpha] \in D_n$ and $e = 0$. Then the ideal α will become principal in K_m when $m \geq n$ is large enough.*

Proof. Let $\mathfrak{P}_{i,n}$ be the prime of K_n that lies above $\mathfrak{P}_i \in S$. Then $\mathfrak{P}_{i,n}\mathcal{O}_{K_m} = \mathfrak{P}_{i,m}^{p^{m-n}}$. We may assume that $\alpha = \prod_i \mathfrak{P}_{i,n}^{s_i}$ for some integers s_i. Then $\alpha\mathcal{O}_{K_m} = \prod_i \mathfrak{P}_{i,n}^{s_i}\mathcal{O}_{K_m} = \prod_i \mathfrak{P}_{i,m}^{s_i p^{(m-n)}}$ for $m \geq n$. On the other hand, the class of $\prod_i \mathfrak{P}_{i,m}^{s_i}$ is contained in B_m, and the size B_m is bounded as $m \to \infty$. Hence $\alpha\mathcal{O}_{K_m}$ becomes principal when m is large enough. □

Let $i_{n,m} : \mathrm{Cl}(K_n) \to \mathrm{Cl}(K_m)$ for $m > n$ be the map induced by inclusion, let $\mathrm{N}_{m,n} : \mathrm{Cl}(K_m) \to \mathrm{Cl}(K_n)$ for $m > n$ be the norm map, and write $H_{n,m} = \mathrm{Ker}\ (i_{n,m})$. Because $\mathrm{N}_{m,n} \circ i_{n,m} = p^{m-n}$, we have that $H_{n,m} \subset A_n$. Put $H_n = \bigcup_{m \geq n} H_{n,m}$. Corollary 3.4 says that $D_0 \subset H_0$, and Corollary 3.5 says that $D_n \subset H_n$ when $e = 0$. Greenberg [Gre76, Proposition 2] proved that the size of A_n is bounded as $n \to \infty$ for the cyclotomic $\mathbb{Z}_p$-extension of a totally real field if and only if $H_n = A_n$ for all n. We have the following similar result.

Proposition 3.6. *Assuming Leopoldt's conjecture holds for p and K, we have that $|A_n|$ is bounded if and only if $H_n = A_n$ for all n for the S-ramified $\mathbb{Z}_p$-extension.*

Proof. First, Iwasawa proved that $|H_n|$ is bounded [Iwa73, Theorem 10 on page 264], so $|A_n|$ is bounded if $A_n = H_n$ for all n.

Now, assume $|A_n|$ is bounded. Let L'_n be the maximal pro-p abelian extension of K_n that is unramified over K_n. Then $A_n \cong \mathrm{Gal}(L'_n/K_n)$. When $m \geq n \geq e$, the field extension K_m/K_n is totally ramified at primes above S. This implies that $K_m \cap L'_n = K_n$, and therefore the restriction map from $\mathrm{Gal}(L'_m/K_m) \to \mathrm{Gal}(L'_n/K_n)$ is surjective. Hence, the norm map $\mathrm{N}_{m,n} : A_m \to A_n$ is surjective for $m \geq n \geq e$. Since we are assuming $|A_n|$ is bounded as $n \to \infty$, we have $\mathrm{N}_{m,n}$ is an isomorphism when $m \geq n \geq n_0$ for some sufficiently large integer n_0.

Take $c \in A_n$. Define $c_r := i_{n,r}(c)$ where $r \geq n_0$. Take m large enough such that $c_r^{p^{m-r}} = 1$. Then $\mathrm{N}_{m,r}(i_{r,m}(c_r)) = c_r^{p^{m-r}} = 1$. We know $\mathrm{Ker}\, \mathrm{N}_{m,r} = 1$ since $m \geq r \geq n_0$, which implies that $i_{n,m}(c) = i_{r,m}(c_r) = 1$. Hence $c \in H_n$ by definition. □

To simplify our upcoming arguments, we make one more assumption throughout the rest of the paper:

(4) Assume that K_∞/K is totally ramified at all primes in S.

In other words, we further assume $e = 0$. One big advantage of assumption (4) is that it implies $D_n \subset B_n$.

4. Analogues in Special Cases

We keep assumptions (2), (3), and (4) in this section. In other words, we assume that each prime above odd prime p in K^+ is split in K and the S-ramified $\mathbb{Z}_p$-extension K_∞/K is totally ramified at all primes in S.

Let us recall some well-known formulas for the order of the so called *ambiguous* and *strong ambiguous class groups*, which plays more important role in the later sections. Let L/F be a cyclic extension of number fields, and σ a generator of the Galois group $\mathrm{Gal}(L/F)$. We call an ideal class $[c] \in \mathrm{Cl}(L)$ an ambiguous class if $[c]^\sigma = [c]$. We call an ideal class $[c] \in \mathrm{Cl}(L)$ a strongly ambiguous class if $c^\sigma = c$, that is

$$\mathrm{Am}(L/F) := \{[c] \in \mathrm{Cl}(L) \,|\, [c]^\sigma = [c]\}$$

and

$$\mathrm{Am}(L/F)_{st} := \{[c] \in \mathrm{Cl}(L) \,|\, c^\sigma = c\}.$$

In other words, the group $\mathrm{Am}(L/F)$ consists of ideal classes that are fixed by the Galois group $\mathrm{Gal}(L/F)$, and the group $\mathrm{Am}_{st}(L/F)$ consists of ideal classes that contain an ideal that is fixed by the Galois group $\mathrm{Gal}(L/F)$. The orders of these groups are given by Chevalley [Che33]:

$$\begin{aligned} |\mathrm{Am}(L/F)| &= \frac{h_F \prod_v e_v}{[L:F][\mathcal{O}_F^\times : \mathcal{O}_F^\times \cap \mathrm{N}(L^\times)]} \\ |\mathrm{Am}(L/F)_{st}| &= \frac{h_F \prod_v e_v}{[L:F][\mathcal{O}_F^\times : \mathrm{N}(\mathcal{O}_L^\times)]} \end{aligned} \tag{5}$$

where the product is taken over all places v of F, e_v is the ramification degree of v in L/F, and h_F is the class number of F. For any abelian group M, we write $M[p^\infty] = \{x \in M \,|\, p^n x = 0 \text{ for some } n\}$ to denote the p-part of M.

Greenberg's criteria in the inert and completely split cases admit a unified interpretation in terms of genus formulas; see [Ngu18]. We now establish analogous criteria for the S-ramified $\mathbb{Z}_p$-extension of a CM field.

Firstly, we list the case that p remains prime in K^+. Hence there are only two primes $\mathfrak{P}$ and $\tilde{\mathfrak{P}}$ above p in K, and we can take $S = \{\mathfrak{P}\}$.

Theorem 4.1. *Suppose that the odd prime p is inert in $K^+/\mathbb{Q}$ and Leopoldt's conjecture holds. Assume that each prime above p in K^+ is split in K and the unique S-ramified $\mathbb{Z}_p$-extension K_∞/K is totally ramified at all primes in S. With the same notation as before, the following statements are equivalent:*

(a) $A_0 = H_0$.

(b) $|A_n|$ *is bounded as* $n \to \infty$.

Remark 4.2. Greenberg's Theorem 1 in [Gre76] states a similar criterion for cyclotomic $\mathbb{Z}_p$-extension of a totally real field when p remains prime.

The above theorem is a special case of [Min86, Prop, 1.B.]. As the reference of [Min86] is difficult to find, we include a proof here for completeness.

Proof. By Proposition 3.6, if $|A_n|$ is bounded as $n \to \infty$, then $A_n = H_n$ for every n. In particular, $A_0 = H_0$.

Conversely, assume that $A_0 = H_0$. Let K_n be the n-th layer of the S-ramified $\mathbb{Z}_p$-extension K_∞/K. By assumption, there is only one prime ramified in K_m/K_n for $m > n$. Every unit in $\mathcal{O}_{K_n}^\times$ is a local norm at each unramified prime, since the norm map on local units is surjective for an unramified extension. By global reciprocity and the product formula for local Artin symbols, it is then also a local norm at the unique ramified prime. Hence, by the Hasse norm theorem,

$$\mathcal{O}_{K_n}^\times = \mathcal{O}_{K_n}^\times \cap \mathrm{N}_{m,n}(K_m^\times),$$

where $\mathrm{N}_{m,n}$ denotes the norm map from K_m to K_n. Therefore, Chevalley's formula (5) gives

$$|A_m^{\mathrm{Gal}(K_m/K_n)}| = |A_n|.$$

The same argument as in the proof of Proposition 3.6 shows that the norm map $\mathrm{N}_{m,n}\colon A_m \to A_n$ is surjective. Let σ be a topological generator of $\Gamma = \mathrm{Gal}(K_\infty/K)$ and put $\omega_n = \sigma^{p^n} - 1$. Since $\omega_n A_m \subseteq \ker \mathrm{N}_{m,n}$, the norm map induces a surjection $A_m/\omega_n A_m \twoheadrightarrow A_n$. Moreover, $|A_m/\omega_n A_m| = |\operatorname{coker}(\omega_n)| = |\ker(\omega_n)| = |A_m^{\mathrm{Gal}(K_m/K_n)}| = |A_n|$. Hence $A_m/\omega_n A_m \cong A_n$. Let $X := \varprojlim_m A_m$. Passing to the inverse limit, we obtain $A_n \cong X/\omega_n X$.

Let F be the maximal finite $\mathbb{Z}_p[[\Gamma]]$-submodule of X. By the algebraic argument preceding [Jau19, Theorem 5], we have

$$H_n = \frac{F + \omega_n X}{\omega_n X}.$$

In particular, since $A_0 = H_0$,

$$\frac{X}{\omega_0 X} = \frac{F + \omega_0 X}{\omega_0 X}.$$

Therefore, $X = F + \omega_0 X$. Passing to the quotient by F, we obtain $X/F = \omega_0(X/F)$. Since $\omega_0 = \sigma - 1$ belongs to the Jacobson radical of $\mathbb{Z}_p[[\Gamma]]$, Nakayama's lemma implies that $X/F = 0$. Thus $X = F$, and hence X is finite. It follows that $|A_n|$ is bounded as $n \to \infty$. □

Now , we consider the case that p splits completely in K^+. Goto [Got06] studies this case for an abelian CM field K, but here we do not need to assume K is abelian.

Theorem 4.3. *Assume that p splits completely in K^+ and Leopoldt's conjecture holds for K. Assume that each prime above p in K^+ is split in K and the unique S-ramified $\mathbb{Z}_p$-extension K_∞/K is totally ramified at all primes in S. The following two statements are equivalent:*

(a) $B_n = D_n$ for all sufficiently large n.

(b) $|A_n|$ is bounded as $n \to \infty$.

Remark 4.4. Greenberg [Gre76, Theorem 2] states a similar criterion for the cyclotomic $\mathbb{Z}_p$-extension of a totally real field when p splits completely. The method of proof is also similar to Greenberg's.

Proof of Theorem 4.3. Assume that $B_n = D_n$ for all sufficiently large n. Since $\mathrm{N}_{m,n} : D_m \to D_n$ is surjective and B_n is bounded by Proposition 3.2, we know that $\mathrm{N}_{m,n} : B_m \to B_n$ is isomorphism for all $m \geq n \geq n_0$ for some n_0. Let Ker $(\mathrm{N}_{m,n})$ be the kernel of the map $\mathrm{N}_{m,n} : A_m \to A_n$. Then Ker $(\mathrm{N}_{m,n}) \cap B_m = 1$ for all $m \geq n \geq n_0$ for some n_0. View Ker $(\mathrm{N}_{m,n})$ as an abelian p-group with the usual action of $\mathrm{Gal}(K_m/K)$. By the general theory of group actions, the fixed point of a nontrivial abelian p-group by a p-group is nontrivial. The fixed point of Ker $(\mathrm{N}_{m,n})$ by $\mathrm{Gal}(K_m/K)$ is Ker $(\mathrm{N}_{m,n}) \cap B_m = 1$. Hence Ker $(\mathrm{N}_{m,n}) = 1$, and thus $\mathrm{N}_{m,n} : A_m \to A_n$ is an isomorphism when $m \geq n \geq n_0$. This implies that $|A_n|$ is bounded.

Now assume that $|A_n|$ is bounded as $n \to \infty$. We will prove that

$$B_n = \mathrm{Am}(K_n/K)[p^\infty] = \mathrm{Am}_{st}(K_n/K)[p^\infty]$$

when n is large enough. Recall that $\mathrm{Am}_{st}(K_n/K)[p^\infty] = i_{0,n}(A_0)D_n$ and $i_{0,n}(A_0)$ will become trivial when n is large enough by Proposition 3.6. Therefore, $B_n = \mathrm{Am}_{st}(K_n/K)[p^\infty]$ implies $B_n = D_n$ when n is large enough.

Since $|A_n|$ is bounded, reverse the argument in the first paragraph to get $\mathrm{N}_{m,n} : B_m \to B_n$ is an isomorphism when $m \geq n \geq n_0$ for some n_0. Let $c \in B_n$ and take $c' \in B_m$ such that $\mathrm{N}_{m,n}(c') = c$. Let J be an ideal of K_m such that $[J] = c'$ and let $I = \mathrm{N}_{m,n}(J)$. Then $[I] = c$. Let $J^{\sigma-1} = (\beta)$ and $I^{\sigma-1} = (\alpha)$, where $\beta \in K_m^\times$ and $\alpha = \mathrm{N}_{m,n}(\beta)$. Set $\varepsilon = \mathrm{N}_{m,0}(\beta) = \mathrm{N}_{n,0}(\alpha)$. Then $\varepsilon \in \mathcal{O}_K^\times$.

Let $K_{n,\mathfrak{P}_i}$ be the localization of K_n at $\mathfrak{P}_i$. We have $K_{\mathfrak{P}_i} \cong \mathbb{Q}_p$ because p splits completely in K. By local class field theory, a local unit in $\mathcal{O}_{K_{\mathfrak{P}_i}}$ sits in $\mathrm{N}_{m,0}(K_{m,\mathfrak{P}_i}^\times)$ if and only if it is a p^m-th power in $K_{\mathfrak{P}_i}^\times$. Hence, ε is a p^m-th power in $K_{\mathfrak{P}_i}^\times$ (see Section 5). Let $\mathcal{P}_i$ be the prime ideal in K^+ below $\mathfrak{P}_i$. Since p splits completely in K, we have $K_{\mathfrak{P}_i} \cong K_{\mathcal{P}_i}^+ \cong K_{\tilde{\mathfrak{P}}_i} \cong \mathbb{Q}_p$.

Recall that the group generated by the roots of unity of K and $\mathcal{O}_{K^+}^\times$ has index 1 or 2 inside $\mathcal{O}_K^\times$ (see Theorem 4.12 in Washington [Was97]). Assume that $\varepsilon^2 = \varepsilon'\varepsilon''$ such that ε' is a root of unity inside K and ε'' is a unit of K^+. Since we assume p splits completely in K, and p splits completely in $\mathbb{Q}(\zeta_n)$ if and only if $p \equiv 1 \pmod{n}$, we have the order of ε' divides $p-1$. Thus ε' is a p^m-th power.

Since ε'' is a unit in K^+, we have ε^2 is a p^m-th power in $K_{\mathfrak{P}_i}^\times$ if and only if ε'' is a p^m-th power in $K_{\mathfrak{P}_i}^\times$ if and only if ε'' is a p^mth power in $(K^+)_{\mathcal{P}_i}^\times$ if and only if ε'' is a p^m-th power in $K_{\tilde{\mathfrak{P}}_i}^\times$ if and only if ε^2 is a p^m-th power in $K_{\tilde{\mathfrak{P}}_i}^\times$. In other words, ε^2 is a p^m-th power after localization at any prime above p. Let $x, y \in \mathbb{Z}$

such that $2x + p^m y = 1$, so that $\varepsilon = \varepsilon^{2x+p^m y}$ is a p^m-th power after localization at any prime above p. By Leopoldt's conjecture and enlarging m if necessary, we may assume $\varepsilon = \eta^{p^n}$ for some $\eta \in \mathcal{O}_K^\times$.

Since $\mathrm{N}_{n,0}(\alpha\eta^{-1}) = 1$, there exists $\gamma \in K_n$ such that $\alpha\eta^{-1} = \gamma^{\sigma-1}$. Thus

$$I^{\sigma-1} = (\alpha) = (\alpha\eta^{-1}) = (\gamma^{\sigma-1})$$

So the ideal class $c \in B_n$ contains a fractional ideal $I(\gamma)^{-1}$ that is fixed by $\mathrm{Gal}(K_n/K)$. Hence $c \in \mathrm{Am}_{st}(K_n/K)[p^\infty]$. Therefore, $B_n = \mathrm{Am}_{st}(K_n/K)[p^\infty]$.

□

5. The Ambiguous Class Groups

Let p be an odd prime, and K a CM field satisfying Leopoldt's conjecture with maximal real subfield K^+. Further, suppose that p splits completely in $K/\mathbb{Q}$. In this section, we compare the S-ramified $\mathbb{Z}_p$-extension K_∞/K of Theorem 2.1 and the cyclotomic $\mathbb{Z}_p$-extension K_∞^+/K^+ for the totally real field K^+ by computing a certain norm index. Fukuda and Komatsu [FK86a] give a numerical criterion to determine if $\lambda = 0$ for the cyclotomic $\mathbb{Z}_p$-extension of a real quadratic field. We will give an analogous result for the S-ramified $\mathbb{Z}_p$-extension K_∞/K of an imaginary biquadratic field.

Let K_n be the n-th layer of the S-ramified $\mathbb{Z}_p$-extension K_∞/K, and let K_n^+ be the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension of K_∞^+/K^+. We assume that any ramified primes in K_∞/K or K_∞^+/K^+ are totally ramified. Let N_n be the norm map from K_n to K or from K_n^+ to K^+.

Proposition 5.1. *With the above set up,*

$$\mathcal{O}_K^\times/(N_n(K_n^\times) \cap \mathcal{O}_K^\times) \cong \mathcal{O}_{K^+}^\times/(N_n((K_n^+)^\times) \cap \mathcal{O}_{K^+}^\times)$$

Proof. Since both sides are finite abelian p-groups, it suffices to show that

$$\frac{\mathcal{O}_K^\times \otimes \mathbb{Z}_p}{(\mathrm{N}_n(K_n^\times) \cap \mathcal{O}_K^\times) \otimes \mathbb{Z}_p} \cong \frac{\mathcal{O}_{K^+}^\times \otimes \mathbb{Z}_p}{(\mathrm{N}_n((K_n^+)^\times) \cap \mathcal{O}_{K^+}^\times) \otimes \mathbb{Z}_p}. \tag{6}$$

Recall that $S^+ = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_s\}$ is the set of primes of K^+ above p. Write $\mathcal{P}_i\mathcal{O}_K = \mathfrak{P}_i\widetilde{\mathfrak{P}}_i$ for $1 \le i \le s$, where $\widetilde{\mathfrak{P}}_i$ is the complex conjugate of $\mathfrak{P}_i$, and set $S = \{\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_s\}$.

Consider the diagonal localization map

$$\mathcal{O}_K^\times \longrightarrow \prod_{i=1}^s \mathcal{O}_{K_{\mathfrak{P}_i}}^\times \times \prod_{i=1}^s \mathcal{O}_{K_{\widetilde{\mathfrak{P}}_i}}^\times.$$

Leopoldt's conjecture tells us we can identify the closure of its image of above map with $\mathcal{O}_K^\times \otimes \mathbb{Z}_p$.

Let $W(K)$ denote the group of roots of unity in K. Since the odd prime p splits completely in K, the order of $W(K)$ is coprime to p. Moreover, since $[\mathcal{O}_K^\times : W(K)\mathcal{O}_{K^+}^\times] \le 2$, the inclusion $\mathcal{O}_{K^+}^\times \hookrightarrow \mathcal{O}_K^\times$ induces an isomorphism

$$\mathcal{O}_{K^+}^\times \otimes \mathbb{Z}_p \cong \mathcal{O}_K^\times \otimes \mathbb{Z}_p.$$

In particular, complex conjugation τ acts trivially on $\mathcal{O}_K^\times \otimes \mathbb{Z}_p$.

We now compare the denominators in (6). Since p splits completely in K, we have $K_{\mathfrak{P}_i} \cong K_{\mathcal{P}_i}^+ \cong \mathbb{Q}_p$. Since K_n/K is totally ramified at the primes in S, a local unit at $\mathfrak{P}_i$ is a norm from $(K_n)_{\mathfrak{P}_i}$ if and only if it

is a p^n-th power. At the primes in S^τ, the extension K_n/K is unramified, so the norm map on local units is surjective. Hence, by the Hasse norm theorem and local reciprocity,

$$(\mathrm{N}_n(K_n^\times)\cap\mathcal{O}_K^\times)\otimes\mathbb{Z}_p\cong(\mathcal{O}_K^\times\otimes\mathbb{Z}_p)\cap\left(\prod_{i=1}^s(\mathcal{O}_{K_{\mathfrak{P}_i}}^\times)^{p^n}\times\prod_{i=1}^s\mathcal{O}_{K_{\widetilde{\mathfrak{P}}_i}}^\times\right).$$

Similarly, since every prime of K^+ above p is totally ramified in the cyclotomic $\mathbb{Z}_p$-extension,

$$(\mathrm{N}_n((K_n^+)^\times)\cap\mathcal{O}_{K^+}^\times)\otimes\mathbb{Z}_p\cong(\mathcal{O}_{K^+}^\times\otimes\mathbb{Z}_p)\cap\prod_{i=1}^s(\mathcal{O}_{K^+_{\mathcal{P}_i}}^\times)^{p^n}=(\mathcal{O}_K^\times\otimes\mathbb{Z}_p)\cap\left(\prod_{i=1}^s(\mathcal{O}_{K_{\mathfrak{P}_i}}^\times)^{p^n}\times\prod_{i=1}^s(\mathcal{O}_{K_{\widetilde{\mathfrak{P}}_i}}^\times)^{p^n}\right).$$

The last equality uses the identifications $\mathcal{O}_{K^+_{\mathcal{P}_i}}^\times\cong\mathcal{O}_{K_{\mathfrak{P}_i}}^\times\cong\mathcal{O}_{K_{\widetilde{\mathfrak{P}}_i}}^\times$ and $\mathcal{O}_{K^+}^\times\otimes\mathbb{Z}_p\cong\mathcal{O}_K^\times\otimes\mathbb{Z}_p$. Since complex conjugation acts trivially on $\mathcal{O}_K^\times\otimes\mathbb{Z}_p$, an element of $\mathcal{O}_K^\times\otimes\mathbb{Z}_p$ lies in $(\mathcal{O}_{K_{\mathfrak{P}_i}}^\times)^{p^n}$ if and only if its conjugate component lies in $(\mathcal{O}_{K_{\widetilde{\mathfrak{P}}_i}}^\times)^{p^n}$. Therefore, the two denominators in (6) coincide under the above identification. This proves (6). □

Remark 5.2. Notice that $K_n^+\not\subset K_n$. Proposition 5.1 is interesting because we do not have a direct relation between K_n and K_n^+. Though the extensions K_n/K and K_n^+/K^+ are globally different and unrelated, they are locally similar due to the assumption that p splits completely in $K/\mathbb{Q}$.

Given an extension L/M of number fields, let $\mathrm{Am}_p(L/M)$ denote the p-ambiguous class group, that is

$$\mathrm{Am}_p(L/M)=A(L)^{\mathrm{Gal}(L/M)}$$

where $A(L)$ is the p-class group of L.

Corollary 5.3. *Assume that p splits completely in K and ramified primes in K_∞/K or K_∞^+/K^+ are totally ramified. With the above setup, we have*

$$\frac{|Am_p(K_n/K)|}{|A(K)|}=\frac{|Am_p(K_n^+/K^+)|}{|A(K^+)|}.$$

Proof. Chevalley's formula (5) has that

$$|\mathrm{Am}_p(K_n/K)|=|A(K)|\frac{\prod_{\mathfrak{P}}e(\mathfrak{P}_n/\mathfrak{P})}{[K_n:K][\mathcal{O}_K^\times:\mathrm{N}_n(K_n^\times)\cap\mathcal{O}_K^\times]}$$

where the product ranges over primes $\mathfrak{P}$ of K above p, and $e(\mathfrak{P}_n/\mathfrak{P})$ denotes the ramification index. If $\mathfrak{P}\in S$, then $e(\mathfrak{P}_n/\mathfrak{P})=p^n$. Similarly,

$$|\mathrm{Am}_p(K_n^+/K^+)|=|A(K^+)|\frac{\prod_{\mathcal{P}}e(\mathcal{P}_n/\mathcal{P})}{[K_n^+:K^+][\mathcal{O}_{K^+}^\times:\mathrm{N}_n((K_n^+)^\times)\cap\mathcal{O}_{K^+}^\times]}.$$

Now, if $\mathcal{P}$ ramifies in K_n^+, then $e(\mathcal{P}_n/\mathcal{P})=p^n$, and the number of ramified primes in K_n^+/K^+ is the same as the number of ramified primes in K_n/K. This together with the previous proposition proves the Corollary. □

5.1. **The S-Ramified $\mathbb{Z}_p$-Extensions of Imaginary Biquadratic Fields.** As an application, we prove results analogous to those of Fukuda Komatsu [FK86a] for the S-ramified extension of an imaginary biquadratic field.

Let $m, d \in \mathbb{Z}^+$ that are squarefree and coprime. Denote $k = \mathbb{Q}(\sqrt{-m})$, $K^+ = \mathbb{Q}(\sqrt{d})$, $K = K^+k$, and ε to be the fundamental unit for K. Suppose that $p > 2$ is a prime that splits completely in K, with $p\mathcal{O}_k = \mathfrak{p}\tilde{\mathfrak{p}}$ and $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}\bar{\mathfrak{P}}$. Suppose that

$$k \subseteq k_1 \subseteq k_2 \subseteq \cdots \subset \bigcup_n k_n = k_\infty$$

is the unique $\mathbb{Z}_p$-extension of k unramified outside $\mathfrak{p}$. Put $K_n = K^+k_n$ and $K_\infty = K^+k_\infty$. Then K_∞/K is the S-ramified $\mathbb{Z}_p$-extension of K, where S is the set of primes above $\mathfrak{p}$ in K.

Denote $\mathcal{O}_n^\times$ to be the units of $\mathcal{O}_{K_n}$, and $\mathfrak{P}_n$ to be the prime of $\mathcal{O}_{K_n}$ that lies above $\mathfrak{P}$. Let $\mathrm{N}_{n,m} : K_n \to K_m$ be the norm map, and $\mathrm{N}_n : K_n \to K$ the norm map from K_n to K. For any number field L, we denote h_L to be the class number of L.

The following is an analogue of the main theorem in [FK86a]:

Theorem 5.4. *Suppose $p \nmid h_K$, and that r is the largest positive integer such that*

$$\varepsilon^{p-1} \equiv 1 \pmod{\bar{\mathfrak{P}}^r}.$$

If $N_{r-1}(\mathcal{O}_{r-1}^\times) = \mathcal{O}_0^\times$ then $\mu = \lambda = 0$ for the S-ramified $\mathbb{Z}_p$-extension of K_∞/K defined above.

Remark 5.5. [Yok65, Proposition 1] tells us that $p \nmid h_K$ implies $p \nmid h_{K^+}$ and $p \nmid h_k$. Hence primes ramified in the $\mathbb{Z}_p$-extension k_∞/k are totally ramified. Thus the primes ramified in the $\mathbb{Z}_p$-extension K_∞/K are totally ramified. Since $[K^+ : \mathbb{Q}] = 2$ and $p \geq 3$, primes ramified in the cyclotomic $\mathbb{Z}_p$-extension K_∞^+/K^+ are also totally ramified. Therefore, we may apply Proposition 5.1 in the following proof.

Proof. Suppose that $\mathrm{N}_{r-1}(\mathcal{O}_{r-1}^\times) = \mathcal{O}_0^\times$. Then there is $\beta \in \mathcal{O}_{r-1}^\times$ such that $\mathrm{N}_{r-1}(\beta) = \varepsilon$, and so for any $n \geq r-1$ we have $\varepsilon^{p^{n-r+1}} \in \mathrm{N}_n(\mathcal{O}_n^\times)$. Thus, $|\mathcal{O}_0^\times/\mathrm{N}_n(\mathcal{O}_n^\times)| \leq p^{n-r+1}$. Now, $|\mathcal{O}_0^\times/(\mathrm{N}_n(K_n^\times) \cap \mathcal{O}_0^\times)| = |\mathcal{O}_{K^+}^\times/(\mathrm{N}_n((K_n^+)^\times) \cap \mathcal{O}_{K^+}^\times)|$ by Proposition 5.1. Meanwhile, Fukuda and Komatsu [FK86a, Lemma 2] calculate that $|\mathcal{O}_{K^+}^\times/(\mathrm{N}_n((K_n^+)^\times) \cap \mathcal{O}_{K^+}^\times)| = p^{n-r+1}$, hence $|\mathcal{O}_0^\times/\mathrm{N}_n(\mathcal{O}_n^\times)| \leq p^{n-r+1} = |\mathcal{O}_0^\times/(\mathrm{N}_n(K_n^\times) \cap \mathcal{O}_0^\times)| \leq |\mathcal{O}_0^\times/\mathrm{N}_n(\mathcal{O}_n^\times)|$. By Chevalley's formula, $B_n = D_n$ for all $n \geq r-1$ so that $\mu = \lambda = 0$ by Theorem 4.3. □

Lemma 5.6. *Suppose that p splits in K. Let $\mathcal{P}$ be a prime above p in K^+. Let $\mathfrak{P}$ be a prime above $\mathcal{P}$ in K, ε_K the fundamental unit of $\mathcal{O}_K^\times$, and ε_{K^+} the fundamental unit of K^+. Then*

$$\varepsilon_K^{p-1} \equiv 1 \pmod{\mathfrak{P}^n} \qquad \Longleftrightarrow \qquad \varepsilon_{K^+}^{p-1} \equiv 1 \pmod{\mathcal{P}^n}.$$

Proof. Let $W(K)$ be the group of roots of unity in K. Since p splits completely in K, we have $\#W(K)$ divides $p-1$. Recall that $[\mathcal{O}_K^\times : W(K)\mathcal{O}_{K^+}^\times] \leq 2$ by [Was97, Theorem 2.13], so $\varepsilon_K^2 = \varepsilon_{K^+}^q\zeta$ for some $q \in \mathbb{Z}$ and $\zeta \in W(K)$. Since $\mathcal{O}_{K^+}^\times \subset \mathcal{O}_K^\times$, we have $\varepsilon_{K^+} = \varepsilon_K^r\eta$ for some $r \in \mathbb{Z}$ and $\eta \in W(K)$. Let $K_{\mathcal{P}}^+$ be the completion of K^+ at $\mathcal{P}$ and $K_{\mathfrak{P}}$ be the completion of K at $\mathfrak{P}$. Since p splits completely in K, we have $K_{\mathcal{P}}^+ \cong K_{\mathfrak{P}} \cong \mathbb{Q}_p$. Then $\varepsilon_{K^+}^{p-1} \equiv 1 \pmod{\mathcal{P}^n}$ if and only if $\varepsilon_{K^+}^{p-1}$ is a p^{n-1}-th power if and only if ε_K^{p-1} is a p^{n-1}-th power if and only if $\varepsilon_K^{p-1} \equiv 1 \pmod{\mathfrak{P}^n}$. □

Corollary 5.7 (Analogue of Lemma 4 in [FK86a])**.** *Suppose that p splits in K. Let ε_K be a fundamental unit of $\mathcal{O}_K^\times$. Suppose that $p \nmid h_K$, and that*

$$\varepsilon_K^{p-1} \equiv 1 \pmod{\bar{\mathfrak{P}}^2} \qquad \textit{but} \qquad \varepsilon_K^{p-1} \not\equiv 1 \pmod{\bar{\mathfrak{P}}^3}.$$

Write $\mathfrak{P}^{h_K} = (\alpha)$, and suppose that

$$\alpha^{p-1} \equiv 1 \pmod{\bar{\mathfrak{P}}} \qquad \textit{but} \qquad \alpha^{p-1} \not\equiv 1 \pmod{\bar{\mathfrak{P}}^2}$$

Then $\mu = \lambda = 0$.

Remark 5.8. There is another similar numerical criterion in this setting, given by Itoh [Ito11, Proposition 3.2], which generalizes the criterion of Fukuda and Komatsu [FK86b, Theorem 1].

Proof. We again follow the proof of Fukuda and Komatsu in [FK86a]. Under our assumptions, Lemma 5.6 implies that $\varepsilon_{K^+}^{p-1} \equiv 1 \pmod{\mathcal{P}^2}$ and $\varepsilon_{K^+}^{p-1} \not\equiv 1 \pmod{\mathcal{P}^3}$. Therefore, [FK86a, Lemma 2] gives $\left[\mathcal{O}_{K^+}^\times : N_1((K_1^+)^\times) \cap \mathcal{O}_{K^+}^\times\right] = 1$. Hence Proposition 5.1 gives $N_1(K_1^\times) \cap \mathcal{O}_K^\times = \mathcal{O}_K^\times$. Therefore,

$$[B_1 : D_1] = [\mathrm{N}_1(K_1^\times) \cap \mathcal{O}_K^\times : \mathrm{N}_1(\mathcal{O}_{K_1}^\times)] = [\mathcal{O}_K^\times : \mathrm{N}_1(\mathcal{O}_{K_1}^\times)].$$

By Theorem 5.4, we only need to show that $B_1 = D_1$.

By Proposition 1 in [FK86a] combined with Corollary 5.3, we have $|B_1| = p$. Let $\mathfrak{P}_1$ be the prime of K_1 above $\mathfrak{P}$ in K. Then the class of $\mathfrak{P}_1^{h_K}$ generates D_1 (here we are using the assumption that $p \nmid h_K$). We will show that $\mathfrak{P}_1^{h_K}$ is not principal. Indeed, suppose that $\mathfrak{P}_1^{h_K} = (\alpha_1)$. Then

$$\mathrm{N}_1(\alpha_1) = \alpha \varepsilon_K^t$$

for some $t \in \mathbb{Z}$. Since $N_1(\alpha_1)$ is a local norm at $\bar{\mathfrak{P}}$, local reciprocity gives that $\mathrm{N}_1(\alpha_1)^{p-1} \equiv 1 \pmod{\bar{\mathfrak{P}}^2}$. Thus $\alpha^{p-1} \equiv 1 \pmod{\bar{\mathfrak{P}}^2}$, which contradicts our assumptions on ε and α.

□

Remark 5.9. One may also establish analogues of Fukuda and Komatsu's results [FK86a, Lemma 2] and [FK86a, Proposition 1]. Consequently, it is possible to provide a proof without relying on the properties of cyclotomic $\mathbb{Z}_p$-extensions of totally real fields. However, since we have already established comparisons between these two different $\mathbb{Z}_p$-extensions, such as Proposition 5.1 and Corollary 5.3, we would like to take advantage of these results.

6. The Norm Index

In this section, we generalize a result of Ehud de Shalit [Sha89, Theorem 2]. In this subsection, we keep the assumptions that each of the primes above odd prime p in K^+ split in K. Assume Leopoldt's conjecture holds for K and p. Let $K \subset K_1 \subset K_2 \subset \cdots \subset K_\infty$ be the S-ramified $\mathbb{Z}_p$-extension defined by Theorem 2.1. Lemma 3.1 tells us that K_∞/K_e is totally ramified at all primes above S for some integer $e \geq 0$. We fix such e in the section.

Let $\mathcal{O}_n^\times$ be the group of units in the ring of integers $\mathcal{O}_{K_n}$. Let $\mathrm{N}_{m,n}$ be the norm map from K_m to K_n for $m \geq n$. For $n \geq e$, define

$$E_n^0 = \{\beta \in \mathcal{O}_n^\times \mid \mathrm{N}_{n,e}(\beta) = 1\},$$

$$E'_n = \bigcap_{m \geq n} \mathrm{N}_{m,n}(E^0_m).$$

Hence, $E'_n \subset E^0_n$. Recall A_n is the p-primary part of the class group of K_n.

Theorem 6.1. *Assume each of the primes above odd prime p in K^+ split in K.*

(a) When n is sufficiently large, $[E^0_n : E'_n]$ divides $|A_n|$.

(b) When n is sufficiently large, every element $e_n \in E'_n$ appears in a norm coherent sequence $(e_m) \in \varprojlim \mathcal{O}^\times_n$.

Proof of (a). Recall the equation (5) of ambiguous class groups in the previous section.

$$|\mathrm{Am}(L/F)| = \frac{h_F \prod_v e_v}{[L:F][\mathcal{O}^\times_F : \mathcal{O}^\times_F \cap \mathrm{N}(L^\times)]}$$
$$|\mathrm{Am}(L/F)_{st}| = \frac{h_F \prod_v e_v}{[L:F][\mathcal{O}^\times_F : \mathrm{N}(\mathcal{O}^\times_L)]}$$

We consider the p-primary part of $\mathrm{Am}(K_m/K_n)$ and $\mathrm{Am}(K_m/K_n)_{st}$ for $m \geq n$. By definition of ambiguous class group, we have $\mathrm{Am}(K_m/K_n)[p^\infty] = A_m^{\mathrm{Gal}(K_m/K_n)}$. Recall that $i_{n,m} : \mathrm{Cl}(K_n) \to \mathrm{Cl}(K_m)$ for $m > n$ is the map induced by inclusion and D_n is the subgroup of A_n generated by prime ideals above the primes of S. By Lemma 3.1, when $m \geq n \geq e$, K_m/K_n is totally ramified at all primes above S and unramified outside primes in S. By the definition of strong ambiguous class group, we have $\mathrm{Am}(K_m/K_n)_{st}[p^\infty] = i_{n,m}(A_n)D_m$. For $m \geq n \geq e$, we have

$$|A_m^{\mathrm{Gal}(K_m/K_n)}| = \frac{|A_n| \prod_{v \in S} e_v(K_m/K_n)}{[K_m : K_n][\mathcal{O}^\times_n : \mathcal{O}^\times_n \cap \mathrm{N}_{m,n}(K^\times_m)]},$$

$$|i_{e,m}(A_e)D_m| = \frac{|A_e| \prod_{v \in S} e_v(K_m/K_e)}{[K_m : K_e][\mathcal{O}^\times_e : \mathrm{N}_{m,e}(\mathcal{O}^\times_m)]},$$

$$|i_{e,n}(A_e)D_n| = \frac{|A_e| \prod_{v \in S} e_v(K_n/K_e)}{[K_n : K_e][\mathcal{O}^\times_e : \mathrm{N}_{n,e}(\mathcal{O}^\times_n)]}.$$

By Lemma 6.2 in the following, when n is sufficiently large, we have

$$1 = \frac{\prod_{v \in S} e_v(K_m/K_n)}{[K_m : K_n][\mathrm{N}_{n,e}(\mathcal{O}^\times_n) : \mathrm{N}_{m,e}(\mathcal{O}^\times_m)]}.$$

Hence,

$$|A_m^{\mathrm{Gal}(K_m/K_n)}| = \frac{|A_n|[\mathrm{N}_{n,e}(\mathcal{O}^\times_n) : \mathrm{N}_{m,e}(\mathcal{O}^\times_m)]}{[\mathcal{O}^\times_n : \mathcal{O}^\times_n \cap \mathrm{N}_{m,n}(K^\times_m)]}.$$

Consider the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{N}_{m,n}(E^0_m) & \longrightarrow & \mathrm{N}_{m,n}(\mathcal{O}^\times_m) & \xrightarrow{\mathrm{N}_{n,e}} & \mathrm{N}_{m,e}(\mathcal{O}^\times_m) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E^0_n & \longrightarrow & \mathcal{O}^\times_n & \xrightarrow{\mathrm{N}_{n,e}} & \mathrm{N}_{n,e}(\mathcal{O}^\times_n) & \longrightarrow & 0
\end{array}$$

The vertical map is the natural inclusion. By snake lemma,

$$\begin{aligned}[E_n^0 : \mathrm{N}_{m,n}(E_m^0)] =& \frac{[\mathcal{O}_n^\times : \mathrm{N}_{m,n}(\mathcal{O}_m^\times)]}{[\mathrm{N}_{n,e}(\mathcal{O}_n^\times) : \mathrm{N}_{m,e}(\mathcal{O}_m^\times)]} \\ =& \frac{[\mathcal{O}_n^\times : \mathcal{O}_n^\times \cap \mathrm{N}_{m,n}(K_m^\times)][\mathcal{O}_n^\times \cap \mathrm{N}_{m,n}(K_m^\times) : \mathrm{N}_{m,n}(\mathcal{O}_m^\times)]}{[\mathrm{N}_{n,e}(\mathcal{O}_n^\times) : \mathrm{N}_{m,e}(\mathcal{O}_m^\times)]} \\ =& |A_n| \frac{[\mathcal{O}_n^\times \cap \mathrm{N}_{m,n}(K_m^\times) : \mathrm{N}_{m,n}(\mathcal{O}_m^\times)]}{|A_m^{\mathrm{Gal}(K_m/K_n)}|}\end{aligned}$$

By the exact sequence (4) in de Shalit's paper [Sha89, Equation (4)], we know that $[\mathcal{O}_n^\times \cap \mathrm{N}_{m,n}(K_m^\times) : \mathrm{N}_{m,n}(\mathcal{O}_m^\times)]$ divides $|A_m^{\mathrm{Gal}(K_m/K_n)}|$. Hence, $[E_n^0 : E_n']$ divides $|A_n|$. □

Lemma 6.2. *The size $|i_{e,n}(A_e)D_n|$ is stable when n is sufficiently large.*

Proof. We know $\mathrm{N}_{m,n} : D_m \to D_n$ is a surjective map when $m \geq n \geq e$. Since B_n is bounded and $D_n \subset B_n$, we find that $\mathrm{N}_{m,n} : D_m \to D_n$ is isomorphic when n is sufficiently large.

Obviously, $|i_{e,n}(A_e)|$ is a decreasing sequence. When n is sufficiently large, we have $|i_{e,m}(A_e)| = |i_{e,n}(A_e)|$. We have the following exact sequence.

$$0 \to i_{e,n}(A_e)[p^{m-n}] \to i_{e,m}(A_e) \xrightarrow{\mathrm{N}_{m,n}} i_{e,n}(A_e) \to i_{e,n}(A_e)/\mathrm{N}_{m,n}(i_{e,m}(A_e)) \to 0$$

When n is sufficiently large, $|i_{e,n}(A_e)[p^{m-n}]| = |i_{e,n}(A_e)/\mathrm{N}_{m,n}(i_{e,m}(A_e))|$.

Consider the following exact sequence in which n is sufficiently large.

$$0 \to Ker \to i_{e,m}(A_e)D_m \xrightarrow{\mathrm{N}_{m,n}} i_{e,n}(A_e)D_n \to Coker \to 0$$

Here $i_{e,n}(A_e)[p^{m-n}] \subset Ker$ implies $|Ker| \geq |i_{e,n}(A_e)[p^{m-n}]|$. We have

$$\begin{aligned}Coker =& \frac{i_{e,n}(A_e)D_n}{\mathrm{N}_{m,n}(i_{e,m}(A_e)D_m)} = \frac{i_{e,n}(A_e)D_n}{\mathrm{N}_{m,n}(i_{e,m}(A_e))\mathrm{N}_{m,n}(D_m)} \\ =& \frac{i_{e,n}(A_e)D_n}{\mathrm{N}_{m,n}(i_{e,m}(A_e))D_n} = \frac{i_{e,n}(A_e)\mathrm{N}_{m,n}(i_{e,m}(A_e))D_n}{\mathrm{N}_{m,n}(i_{e,m}(A_e))D_n} \\ =& \frac{i_{e,n}(A_e)}{i_{e,n}(A_e) \cap \mathrm{N}_{m,n}(i_{e,m}(A_e))D_n}.\end{aligned}$$

Thus $|Coker| \leq |i_{e,n}(A_e)/\mathrm{N}_{m,n}(i_{e,m}(A_e))|$.

Because we assume n is sufficiently large, we have

$$\begin{aligned}|i_{e,m}(A_e)D_m| - |i_{e,n}(A_e)D_n| =& |Ker| - |Coker| \\ \geq& |i_{e,n}(A_e)[p^{m-n}]| - |i_{e,n}(A_e)/\mathrm{N}_{m,n}(i_{e,m}(A_e))| \\ =& 0.\end{aligned}$$

It implies that $|i_{e,n}(A_e)D_n|$ is an increasing sequence when n is sufficiently large. On the other hand, $i_{e,n}(A_e)D_n \subset B_n$ and $|B_n|$ is bounded. Hence, $|i_{e,n}(A_e)D_n|$ remains stable when n is sufficiently large. □

proof of (b). (b) is an easy consequence of (a). Indeed, we only use the fact that E_n' is of finite index inside E_n^0. First, notice that $E_n' \subset E_n^0 \subset \mathcal{O}_n^\times$ are finitely generated $\mathbb{Z}$-module. By the structure of finitely generated

$\mathbb{Z}$-module, the number of submodule $M \subset E_n^0$ such that the index $[E_n^0 : M] \leq C$ is finite, where C is a fixed constant. We have

$$[E_n^0 : \mathrm{N}_{n+1,n}(E_{n+1}^0)] \leq [E_n^0 : \mathrm{N}_{n+2,n}(E_{n+2}^0)] \leq \cdots \leq [E_n^0 : \mathrm{N}_{m,n}(E_m^0)] \leq [E_n^0 : \mathrm{N}_{m+1,n}(E_{m+1}^0)] \leq \cdots \leq |A_n|$$

Hence, $[E_n^0 : \mathrm{N}_{m,n}(E_m^0)] = [E_n^0 : \mathrm{N}_{m+1,n}(E_{m+1}^0)]$ for all sufficiently large m. Thus, $\mathrm{N}_{m,n}(E_m^0) = E_n'$ for all sufficiently large m. Similarly, $\mathrm{N}_{k,n+1}(E_k^0) = E_{n+1}'$ for all sufficiently large k. Take $m = k$ when both are very large. Then we see $\mathrm{N}_{n+1,n}(E_{n+1}') = E_n'$. The conclusion follows.

□

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